

BOUNDEDNESS IN A QUASILINEAR FULLY PARABOLIC KELLER-SEGEL SYSTEM WITH LOGISTIC SOURCE

QINGSHAN ZHANG AND YUXIANG LI

ABSTRACT. This paper deals with the Neumann boundary value problem for the system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v) + f(u), & x \in \Omega, \ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, \ t > 0 \end{cases}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 1$), where the functions $D(u)$ and $S(u)$ are supposed to be smooth satisfying $D(u) \geq Mu^{-\alpha}$ and $S(u) \leq Mu^\beta$ with $M > 0$, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ for all $u \geq 1$, and the logistic source $f(u)$ is smooth fulfilling $f(0) \geq 0$ as well as $f(u) \leq a - \mu u^\gamma$ with $a \geq 0$, $\mu > 0$ and $\gamma \geq 1$ for all $u \geq 0$. It is shown that if

$$\alpha + 2\beta < \begin{cases} \gamma - 1 + \frac{2}{n}, & \text{for } 1 \leq \gamma < 2, \\ \gamma - 1 + \frac{4}{n+2}, & \text{for } \gamma \geq 2, \end{cases}$$

then for sufficiently smooth initial data the problem possesses a unique global classical solution which is uniformly bounded.

1. INTRODUCTION

In this paper we consider the initial-boundary value problem for the parabolic-parabolic quasilinear chemotaxis system with logistic source

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v) + f(u), & x \in \Omega, \ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, \ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega \end{cases} \quad (1.1)$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) with smooth boundary $\partial\Omega$, where $u = u(x, t)$ denotes the density of bacteria and $v = v(x, t)$ is the concentration of oxygen. $\frac{\partial}{\partial \nu}$ represents differentiation with respect to the outward normal ν on $\partial\Omega$. The initial data $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\theta}(\Omega)$ with $\theta > \max\{2, n\}$ are nonnegative functions. The parameter functions $D(u)$, $S(u)$ with $S(0) = 0$ from $C^2([0, \infty))$ are supposed that

$$D(u) \geq M_1(u+1)^{-\alpha} \quad \text{for all } u \geq 0 \quad (1.2)$$

with $M_1 > 0$ and $\alpha \in \mathbb{R}$,

$$S(u) \leq M_2(u+1)^\beta \quad \text{for all } u \geq 0 \quad (1.3)$$

with $M_2 > 0$ and $\beta \in \mathbb{R}$, as well as $f(u)$ is smooth satisfying $f(0) \geq 0$ and

$$f(u) \leq a - \mu u^\gamma \quad \text{for all } u \geq 0 \quad (1.4)$$

with $a \geq 0$, $\mu > 0$ and $\gamma \geq 1$.

2000 *Mathematics Subject Classification.* 35K59, 92C17, 35K55.

Key words and phrases. quasilinear chemotaxis system, logistic source, global solution, boundedness.
Supported in part by National Natural Science Foundation of China (No. 11171063).

In 1970, Keller and Segel [7] proposed the chemotaxis system (1.1) to describe the biased movement of biological cell in response to chemical gradients. Since then the model has attracted significant interest in mathematical biology, and one of the main issues is under what conditions the solutions of (1.1) blow up or exist globally.

When $D(u) = 1$, $S(u) = \chi u$ and $f(u) \equiv 0$, system (1.1) corresponds to the so-called minimal model, which has been extensively studied. It proved that the solutions never blow up if $n = 1$ [13]. In the two-dimensional case, if $\int_{\Omega} u_0 < 4\pi$ the solutions are global and bounded [11], whereas $\int_{\Omega} u_0 > 4\pi$ and in the case $n \geq 3$ the solutions blow up in finite time [4, 21]. In many applications the blow-up phenomena is an extreme case, so a logistic growth restriction of type (1.4) in (1.1) is expected to rule out the possible of blow-up for solutions. When $D(u) = 1$, $S(u) = \chi u$ and $f(u) \leq a - \mu u^2$ in the model (1.1), all solutions are global and bounded provided that $n \leq 2$ or $\mu > \mu_0$ with some $\mu_0 > 0$ in higher dimensions $n \geq 3$ [12, 13, 18]. In the special case $f(u) = u - \mu u^2$, whenever $\frac{\mu}{\chi}$ is suitably large the solution (u, v) stabilizes to the spatially homogeneous steady state $(\frac{1}{\mu}, \frac{1}{\mu})$ as $t \rightarrow \infty$ [22]. Moreover, when $n \geq 3$ there exists at least one global weak solution for any $\mu > 0$ [8]. However, it is unclear whether in higher dimensions $n \geq 3$, the logistic source $f(u)$ with $\gamma = 2$ in the problem (1.1) might be sufficient to rule out blow-up for arbitrarily small $\mu > 0$ [18].

Superlinear logistic growth are not always rule out chemotactic collapse in the Keller-Segel model. The initial-boundary value problem for the related system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \lambda u - \mu u^\gamma, & x \in \Omega, \ t > 0, \\ 0 = \Delta v - m(t) + u, & x \in \Omega, \ t > 0 \end{cases}$$

with $m(t) := \frac{1}{|\Omega|} \int_{\Omega} u(x, t)$ was considered in [20]. It was shown that if $\lambda \in (1, \frac{3}{2} + \frac{1}{2n-2})$ with $n \geq 5$ there exist initial data such that the solutions blow up in finite time.

On the other hand, the volume-filling effect can also prevent blow-up [3, 14]. In the case $D(u) = 1$ and $f(u) \equiv 0$ in (1.1). Horstmann and Winkler [5] proved that if $S(u) \leq K(u+1)^\beta$ with $\beta < \frac{2}{n}$ and some $K > 0$ the solutions are global and bounded, while if $S(u) \geq K(u+1)^\beta$ with $\beta > \frac{2}{n}$ and some $K > 0$ the solutions blow up in finite or infinite time.

As to the Neumann boundary value problem for the associated parabolic-elliptic system

$$\begin{cases} u_t = \nabla \cdot (D(u) \nabla u) - \nabla \cdot (S(u) \nabla v), & x \in \Omega, \ t > 0, \\ 0 = \Delta v - m(t) + u, & x \in \Omega, \ t > 0 \end{cases}$$

with $D(u) \simeq u^{-\alpha}$ and $S(u) \simeq u^\beta$ as $u \simeq \infty$, it was proved that if $\alpha + \beta < \frac{2}{n}$ the solutions are global and bounded, whereas if $\alpha + \beta > \frac{2}{n}$ there exist solutions that become unbounded in finite time [23].

The model (1.1) with $f(u) \equiv 0$ has also been extensively studied [2, 6, 16, 19]. It was shown that

- if $\frac{S(u)}{D(u)} \leq K(u + \varepsilon)^\theta$ with $\theta < \frac{2}{n}$ and $K > 0$ for some $\varepsilon \geq 0$ and for all $u > 0$, then all solutions to (1.1) are global and uniformly bounded [6, 16];
- if $\frac{S(u)}{D(u)} \geq K(u + 1)^\theta$ with $\theta > \frac{2}{n}$ ($n \geq 2$) and $K > 0$ for all $u > 0$, then for any $m > 0$, (1.1) possesses finite-time blow-up solutions with the mass $\int_{\Omega} u_0 = m$ [19].

The exponent $\frac{2}{n}$ seems critical for the finite-time blow-up and global existence properties of (1.1) with $f(u) \equiv 0$.

The fully parabolic Keller-Segel system with logistic source (1.1) was considered in the recent papers [1, 9, 17]. The authors proved that when $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is a bounded *convex* domain with smooth boundary, global bounded classical solutions exist provided that $D(u)$, $S(u)$ and $f(u)$ satisfy (1.2)-(1.4) with $\alpha + \beta \in (0, \frac{2}{n})$. In [9], the authors extended the result to the degenerate case on non-convex domain under the same condition. Connected to the later two results, it is a natural question to ask:

(Q1) *What role does the logistic source play in the system (1.1)?*

Although a partial answer to (Q1) show that for any choice of $\beta < 1$ the logistic damping rule out the occurrence of blow-up for the related special case $f(u) \leq a - \mu u^2$ provided that S satisfies the condition of algebraic growth [1], it still remain to analysis:

(Q2) *Can we provide a explicit condition involving the nonlinear diffusion, nonlinear chemosensitivity and the logistic-growth source to ensure global bounded solutions in the system (1.1) ?*

In the present paper, our purpose is to answer (Q1) and (Q2). Namely, we shall give a general condition on α, β and γ , which guarantees the global existence and boundedness of classical solutions to (1.1) in *non-convex* bounded domains. Our main result is stated as follows.

Theorem 1.1. *Suppose that $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a domain with smooth boundary. Let (1.2)-(1.4) hold with $M_1 > 0$, $M_2 > 0$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $a \geq 0$, $\mu > 0$ and $\gamma \geq 1$. If*

$$\alpha + 2\beta < \begin{cases} \gamma - 1 + \frac{2}{n}, & \text{if } 1 \leq \gamma < 2, \\ \gamma - 1 + \frac{4}{n+2}, & \text{if } \gamma \geq 2, \end{cases}$$

then for any nonnegative $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\theta}(\Omega)$ with $\theta > \max\{2, n\}$, the problem (1.1) admits a unique global bounded classical solution.

Remark 1.1. (i) Theorem 1.1 provides the general admissible parameters for global boundedness in (1.1). The result is valid on the non-convex domain and without any further assumptions on the size of $a \geq 0$ and $\mu > 0$.

(ii) In the case $\beta \leq 0$ or $\gamma > \beta + 1$ for $\beta > 0$ or $\gamma > \beta + 1 - \frac{2(n-2)}{n(n+2)}$ for $\beta \geq 1 + \frac{2(n-2)}{n(n+2)}$, our result extends the recent work [17] and [9] which assert boundedness under the condition $\alpha + \beta < \frac{2}{n}$.

(iii) For the most interesting classical model

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + a - \mu u^\gamma, & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, t > 0, \end{cases} \quad (1.5)$$

our result shows that if $\gamma > 3 - \frac{4}{n+2}$, the model (1.5) possesses a unique global classical solution for arbitrarily small $\mu > 0$ and any bounded smooth domain $\Omega \subset \mathbb{R}^n$. This improves the recent result [24, R5]. In particular, we rule out a chemotactic collapse in model (1.5) with a cubic growth source $f(u) = u(u - b)(1 - u)$ ($0 < b < \frac{1}{2}$) for any biological parameters and any bounded smooth domain $\Omega \subset \mathbb{R}^n$.

2. PRELIMINARIES

To begin with, let us first state one result concerning local well-posedness of the problem (1.1) and its proof can be found in [9, 16–18].

Lemma 2.1. *Let $D(u)$ and $S(u)$ satisfy (1.2) and (1.3), respectively. Suppose that $f(u) \in W_{loc}^{1,\infty}(\mathbb{R})$ satisfying $f(0) \geq 0$, and that $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\theta}(\Omega)$ with $\theta > \max\{2, n\}$ are nonnegative functions. Then there exist $T_{\max} \in (0, \infty]$ and a uniquely determined pair (u, v) of nonnegative functions*

$$\begin{aligned} u &\in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \\ v &\in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \cap L_{loc}^\infty([0, T_{\max}); W^{1,\theta}(\Omega)) \end{aligned}$$

solves (1.1) in the classical sense. In addition, if $T_{\max} < \infty$, then

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\theta}(\Omega)} \rightarrow \infty \quad \text{as } t \nearrow T_{\max}.$$

We next give the following basic estimates in spatial Lebesgue spaces for u and ∇v .

Lemma 2.2. *Suppose that D , S and f satisfy (1.2)-(1.4), respectively. Then there exists $C > 0$ such that the solution of (1.1) fulfils*

$$\|u(\cdot, t)\|_{L^1(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}) \quad (2.1)$$

and

$$\|\nabla v(\cdot, t)\|_{L^s(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}), \quad (2.2)$$

where $s \in [1, \frac{n}{n-1})$. If, in addition, $\gamma \geq 2$, then we have

$$\|\nabla v(\cdot, t)\|_{L^2(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}). \quad (2.3)$$

Proof. Integrating the first equation in (1.1) and using (1.4) gives

$$\frac{d}{dt} \int_{\Omega} u = \int_{\Omega} f(u) \leq a|\Omega| - \mu \int_{\Omega} u^{\gamma} \quad \text{for all } t \in (0, T_{\max}). \quad (2.4)$$

Since $\gamma \geq 1$, we can find $C_1 > 0$ such that $u \leq u^{\gamma} + C_1$ and thus derive that

$$\frac{d}{dt} \int_{\Omega} u = -\mu \int_{\Omega} u + (a + C_1\mu)|\Omega| \quad \text{for all } t \in (0, T_{\max}).$$

This yields

$$\int_{\Omega} u \leq \left\{ \int_{\Omega} u_0, \frac{(a + C_1\mu)|\Omega|}{\mu} \right\} =: C_2 \quad \text{for all } t \in (0, T_{\max}). \quad (2.5)$$

The proof of (2.2) is based on standard regularity arguments for the heat equation; for details, we refer the reader to [5, Lemma 4.1]. Now, we prove (2.3). Using $-2\Delta v$ as a test function for the second equation in (1.1), integrating over Ω and applying Young's inequality, we have

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |\nabla v|^2 + 2 \int_{\Omega} |\Delta v|^2 + 2 \int_{\Omega} |\nabla v|^2 &= -2 \int_{\Omega} u \Delta v \\ &\leq \frac{1}{2} \int_{\Omega} u^2 + 2 \int_{\Omega} |\Delta v|^2 \end{aligned} \quad (2.6)$$

for all $t \in (0, T_{\max})$. Combining (2.4) and (2.6), we deduce that

$$\frac{d}{dt} \left(\int_{\Omega} u + 2\mu \int_{\Omega} |\nabla v|^2 \right) + 4\mu \int_{\Omega} |\nabla v|^2 \leq \mu \int_{\Omega} u^2 + a|\Omega| - \mu \int_{\Omega} u^{\gamma} \quad (2.7)$$

for all $t \in (0, T_{\max})$. Adding the term $2 \int_{\Omega} u$ in both side of (2.7) and using (2.5) yields

$$\frac{d}{dt} \left(\int_{\Omega} u + 2\mu \int_{\Omega} |\nabla v|^2 \right) + 2 \left(\int_{\Omega} u + 2\mu \int_{\Omega} |\nabla v|^2 \right) \leq -\mu \int_{\Omega} (u^{\gamma} - u^2) + a|\Omega| + 2C_2$$

for all $t \in (0, T_{\max})$. This, along with $\gamma \geq 2$ by our assumption, gives $C_3 > 0$ such that

$$\frac{d}{dt} \left(\int_{\Omega} u + 2\mu \int_{\Omega} |\nabla v|^2 \right) + 2 \left(\int_{\Omega} u + 2\mu \int_{\Omega} |\nabla v|^2 \right) \leq C_3,$$

for all $t \in (0, T_{\max})$, whereby the proof is completed. \square

For $p \geq 1$, $q \geq 1$ and $s \geq 1$, we define

$$\theta_1 := \theta_1(p, q) = \frac{2(p + \gamma - 1)}{\gamma + 1 - \alpha - 2\beta}, \quad (2.8)$$

$$\theta_2 := \theta_2(p, q) = \frac{2(q - 1)(p + \gamma - 1)}{p + \gamma - 3}, \quad (2.9)$$

$$\kappa_i := \kappa_i(p, q; s) = \frac{\frac{q}{s} - \frac{q}{\theta_i}}{\frac{q}{s} - (\frac{1}{2} - \frac{1}{n})} \quad (2.10)$$

and

$$f_i(p, q; s) := \frac{\theta_i}{q} \kappa_i(p, q; s) = \frac{\frac{\theta_i}{s} - 1}{\frac{q}{s} - (\frac{1}{2} - \frac{1}{n})}$$

for $i = 1, 2$. Now let state the following results which will be needed in the proof of the boundedness of global solutions. The main idea of the proof is similar to the strategy introduced in [2]. Since a new parameter γ is involved, we prefer to give enough details for the convenience of the reader.

Lemma 2.3. *Let $n \geq 2$, $\gamma \geq 1$, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. Then for sufficiently large $p > 1$,*

(i) if $\alpha + 2\beta < \gamma - 1 + \frac{2}{n}$, we can choose $q > 1$ such that

$$\kappa_i(p, q; \frac{n}{n-1}) \in (0, 1) \quad \text{and} \quad f_i(p, q; \frac{n}{n-1}) < 2 \quad \text{for } i = 1, 2; \quad (2.11)$$

(ii) if $\alpha + 2\beta < \gamma - 1 + \frac{4}{n+2}$, there exists $q > 1$ fulfilling

$$\kappa_i(p, q; 2) \in (0, 1) \quad \text{and} \quad f_i(p, q; 2) < 2 \quad \text{for } i = 1, 2. \quad (2.12)$$

Remark 2.1. Since in the case $1 \leq \gamma < 2$, we only obtain $\|\nabla v\|_{L^s(\Omega)} \leq C$ with $s \in [1, \frac{n}{n-1})$ according to Lemma 2.2. We need to remark here if (2.11) is true, by continuity argument we can choose s sufficiently close to $\frac{n}{n-1}$ satisfies $\kappa_i(p, q; s) \in (0, 1)$ and $f_i(p, q; s) < 2$ ($i = 1, 2$) when p and q are fixed. Hence it is enough to focus on the case $s = \frac{n}{n-1}$.

Proof of Lemma 2.3. We first claim that for $i = 1, 2$ if

$$\theta_i > s \quad \text{and} \quad q > \frac{\theta_i}{2} - \frac{s}{n}, \quad (2.13)$$

then

$$\kappa_i(p, q; s) \in (0, 1) \quad \text{and} \quad f_i(p, q; s) < 2.$$

Indeed, a direct computation shows that the first inequality in (2.13) is equivalent to $\kappa_i(p, q; s) > 0$. On the other hand, $\kappa_i(p, q; s) < 1$ is equivalent to

$$q > \frac{\theta_i}{2} - \frac{\theta_i}{n},$$

which is weaker than the second inequality in (2.13) because of $\theta_i > s$. Moreover, we have $f_i(p, q; s) < 2$ if and only if

$$\frac{\theta_i}{s} - 1 < \frac{2q}{s} - 2(\frac{1}{2} - \frac{1}{n}),$$

which is true due to the second inequality in (2.13). The claim is proved.

Now we consider the case $s = \frac{n}{n-1}$. Note that (2.13) is fulfilled for θ_1 and θ_2 if

$$p > \frac{n(\gamma + 1 - \alpha - 2\beta)}{2(n-1)} - (\gamma - 1), \quad q > \frac{n}{2(n-1)} + 1 \quad (2.14)$$

and

$$\frac{p + \gamma - 1}{\gamma + 1 - \alpha - 2\beta} - \frac{1}{n-1} < q < \frac{n}{2(n-1)}p + \frac{n}{2(n-1)}(\gamma - 1) - \frac{1}{n-1}. \quad (2.15)$$

One can easily remark that q exists if

$$\frac{p + \gamma - 1}{\gamma + 1 - \alpha - 2\beta} < \frac{n}{2(n-1)}p + \frac{n}{2(n-1)}(\gamma - 1),$$

which can be achieved when $\alpha + 2\beta < \gamma - 1 + \frac{2}{n}$. Fix

$$p_0 := \max \left\{ 1, 3 - \gamma, 2 - \alpha - 2\beta, \frac{3n(\gamma + 1 - \alpha - 2\beta)}{2(n-1)} - (\gamma - 1) \right\}.$$

Then, for any $p > p_0$, we can choose q such that p and q satisfy (2.14) and (2.15). Hence we have (2.11).

When $s = 2$, (2.13) is satisfied for θ_1 and θ_2 if

$$p > 2 - \alpha - 2\beta, \quad q > 2 \quad (2.16)$$

and

$$\frac{p + \gamma - 1}{\gamma + 1 - \alpha - 2\beta} - \frac{2}{n} < q < \frac{n+2}{2n}p + \frac{n+2}{2n}(\gamma - 1) - \frac{2}{n}. \quad (2.17)$$

Set

$$\bar{p}_0 := \max \left\{ 1, 3 - \gamma, 2 - \alpha - 2\beta, \frac{(2n+2)(\gamma+1-\alpha-2\beta)}{n} - (\gamma-1) \right\}.$$

Following in the same way as in (2.11), we have for arbitrary $p > \bar{p}_0$, there exists q such that p and q fulfill (2.16) and (2.17). This completes the proof. \square

3. PROOF OF THEOREM 1.1

According to test-function arguments and interpolation arguments along with the basic estimates in Lemma 2.2, we have the boundedness of $\int_{\Omega} u^p$ with $p \geq 1$.

Lemma 3.1. *Suppose that $\Omega \subset \mathbb{R}^n$ ($n \geq 2$). Let (1.2)-(1.4) hold with $M_1 > 0$, $M_2 > 0$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $a \geq 0$, $\mu > 0$ and $\gamma \geq 1$. Assume that*

$$\alpha + 2\beta < \begin{cases} \gamma - 1 + \frac{2}{n}, & \text{if } 1 \leq \gamma < 2, \\ \gamma - 1 + \frac{4}{n+2}, & \text{if } \gamma \geq 2. \end{cases}$$

Then for all $p \in [1, \infty)$ there exists $C > 0$ such that

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}).$$

Proof. Multiplying the first equation in (1.1) by the test function $(u+1)^{p-1}$, integrating by parts and using (1.4), we obtain

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\Omega} (u+1)^p + (p-1) \int_{\Omega} (u+1)^{p-2} D(u) |\nabla u|^2 + \frac{\mu}{2^\gamma} \int_{\Omega} (u+1)^{p+\gamma-1} \\ & \leq (p-1) \int_{\Omega} (u+1)^{p-2} S(u) \nabla u \cdot \nabla v + (a+\mu) \int_{\Omega} (u+1)^{p-1} \quad \text{for all } t \in (0, T_{\max}) \end{aligned} \quad (3.1)$$

Here, by (1.2),

$$(p-1) \int_{\Omega} (u+1)^{p-2} D(u) |\nabla u|^2 \geq M_1 (p-1) \int_{\Omega} (u+1)^{p-\alpha-2} |\nabla u|^2 \quad \text{for all } t \in (0, T_{\max}), \quad (3.2)$$

from (1.3) and Young's inequality,

$$\begin{aligned} & (p-1) \int_{\Omega} (u+1)^{p-2} S(u) \nabla u \cdot \nabla v \\ & \leq \frac{M_1(p-1)}{2} \int_{\Omega} (u+1)^{p-\alpha-2} |\nabla u|^2 + \frac{p-1}{2M_1} \int_{\Omega} (u+1)^{p+\alpha-2} S^2(u) |\nabla v|^2 \\ & \leq \frac{M_1(p-1)}{2} \int_{\Omega} (u+1)^{p-\alpha-2} |\nabla u|^2 + \frac{M_2^2(p-1)}{2M_1} \int_{\Omega} (u+1)^{p+\alpha+2\beta-2} |\nabla v|^2 \end{aligned} \quad (3.3)$$

for all $t \in (0, T_{\max})$, and again using Young's inequality,

$$(a+\mu) \int_{\Omega} (u+1)^{p-1} \leq \frac{\mu}{2^{\gamma+1}} \int_{\Omega} (u+1)^{p+\gamma-1} + C_1 \quad \text{for all } t \in (0, T_{\max}) \quad (3.4)$$

with $C_1 > 0$. Inserting (3.2)-(3.4) into (3.1) yields

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\Omega} (u+1)^p + \frac{2M_1(p-1)}{(p-\alpha)^2} \int_{\Omega} |\nabla(u+1)^{\frac{p-\alpha}{2}}|^2 + \frac{\mu}{2^{\gamma+1}} \int_{\Omega} (u+1)^{p+\gamma-1} \\ & \leq \frac{M_2^2(p-1)}{2M_1} \int_{\Omega} (u+1)^{p+\alpha+2\beta-2} |\nabla v|^2 + C_1 \quad \text{for all } t \in (0, T_{\max}). \end{aligned} \quad (3.5)$$

Next, differentiating the second equation in (1.1) and using the identity

$$\frac{1}{2} \Delta |\nabla v|^2 = \nabla(\Delta v) \cdot \nabla v + |D^2 v|^2,$$

where $|D^2 v|^2 = \sum_{1 \leq i, j \leq n} (v_{x_i x_j})^2$, we obtain

$$\frac{1}{2} (|\nabla v|^2)_t = \frac{1}{2} \Delta |\nabla v|^2 - |D^2 v|^2 - |\nabla v|^2 + \nabla u \cdot \nabla v$$

for all $(x, t) \in \Omega \times (0, T_{\max})$. We test this by $(|\nabla v|^2)^{q-1}$ and integrate over Ω to have

$$\begin{aligned} & \frac{1}{2q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} + \frac{q-1}{2} \int_{\Omega} |\nabla v|^{2q-4} |\nabla |\nabla v|^2|^2 + \int_{\Omega} |\nabla v|^{2q} + \int_{\Omega} |\nabla v|^{2q-2} |D^2 v|^2 \\ & = \int_{\Omega} |\nabla v|^{2q-2} \nabla u \cdot \nabla v + \frac{1}{2} \int_{\partial\Omega} |\nabla v|^{2q-2} \frac{\partial |\nabla v|^2}{\partial \nu} d\mathbf{S} \quad \text{for all } t \in (0, T_{\max}). \end{aligned} \quad (3.6)$$

Integrating by parts and twice applying Young's inequality, we estimate

$$\begin{aligned} \int_{\Omega} |\nabla v|^{2q-2} \nabla u \cdot \nabla v &= -(q-1) \int_{\Omega} u |\nabla v|^{2q-4} \nabla |\nabla v|^2 \cdot \nabla v - \int_{\Omega} u |\nabla v|^{2q-2} \Delta v \\ &\leq \frac{q-1}{12} \int_{\Omega} |\nabla v|^{2q-4} |\nabla |\nabla v|^2|^2 + 3(q-1) \int_{\Omega} u^2 |\nabla v|^{2q-2} \\ &\quad + \int_{\Omega} |\nabla v|^{2q-2} |D^2 v|^2 + \frac{n}{4} \int_{\Omega} u^2 |\nabla v|^{2q-2} \end{aligned} \quad (3.7)$$

for all $t \in (0, T_{\max})$, where we have used that $\frac{1}{n} |\Delta v|^2 \leq |D^2 v|^2$. As for the second term on the right-hand side of (3.6), we use the inequality [10, Lemma 4.2]

$$\frac{\partial |\nabla w|^2}{\partial \nu} \leq 2k |\nabla w|^2 \quad \text{on } \partial\Omega,$$

with $k = k(\Omega) > 0$ is an upper bound of the curvature of $\partial\Omega$, and apply the trace inequality [15, Remark 52.9]

$$\|w\|_{L^2(\partial\Omega)} \leq \eta \|\nabla w\|_{L^2(\Omega)} + C(\eta) \|w\|_{L^2(\Omega)}$$

to deduce that

$$\begin{aligned} \frac{1}{2} \int_{\partial\Omega} |\nabla v|^{2q-2} \frac{\partial |\nabla v|^2}{\partial \nu} d\mathbf{S} &= \frac{1}{2q} \int_{\partial\Omega} \frac{\partial (|\nabla v|^q)^2}{\partial \nu} d\mathbf{S} \\ &\leq \frac{k}{q} \| |\nabla v|^q \|_{L^2(\partial\Omega)}^2 \\ &\leq \frac{q-1}{3q^2} \|\nabla |\nabla v|^q\|_{L^2(\Omega)}^2 + C_2 \| |\nabla v|^q \|_{L^2(\Omega)}^2 \end{aligned} \quad (3.8)$$

for all $t \in (0, T_{\max})$ with some $C_2 > 0$. From the Gagliardo-Nirenberg inequality, we know that there exist $C_3 > 0$ and $C_4 > 0$ such that

$$\| |\nabla v|^q \|_{L^2(\Omega)}^2 \leq C_3 \|\nabla |\nabla v|^q\|_{L^2(\Omega)}^{2\kappa} \| |\nabla v|^q \|_{L^{\frac{q}{\kappa}}(\Omega)}^{2(1-\kappa)} + C_4 \| |\nabla v|^q \|_{L^{\frac{q}{\kappa}}(\Omega)}^2$$

for all $t \in (0, T_{\max})$, where

$$\kappa = \frac{\frac{q}{s} - \frac{1}{2}}{\frac{q}{s} - \frac{1}{2} + \frac{1}{n}} \in (0, 1)$$

with $s \in [1, \frac{n}{n-1})$. According to (2.2) as well as Young's inequality, we have

$$C_2 \| |\nabla v|^q \|_{L^2(\Omega)}^2 \leq \frac{q-1}{3q^2} \| \nabla |\nabla v|^q \|_{L^2(\Omega)}^2 + C_5 \quad \text{for all } t \in (0, T_{\max}). \quad (3.9)$$

Now collecting (3.6)-(3.13), we get

$$\begin{aligned} & \frac{1}{2q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} + \frac{q-1}{q^2} \| \nabla |\nabla v|^q \|_{L^2(\Omega)}^2 + \int_{\Omega} |\nabla v|^{2q} \\ & \leq \left(3(q-1) + \frac{n}{4} \right) \int_{\Omega} u^2 |\nabla v|^{2q-2} + C_5 \quad \text{for all } t \in (0, T_{\max}). \end{aligned} \quad (3.10)$$

Combining (3.5) with (3.10), it follows that

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{p} \int_{\Omega} (u+1)^p + \frac{1}{2q} \int_{\Omega} |\nabla v|^{2q} \right) \\ & + \frac{2M_1(p-1)}{(p-\alpha)^2} \int_{\Omega} |\nabla(u+1)^{\frac{p-\alpha}{2}}|^2 + \frac{q-1}{q^2} \| \nabla |\nabla v|^q \|_{L^2(\Omega)}^2 \\ & + \frac{\mu}{2^{\gamma+1}} \int_{\Omega} (u+1)^{p+\gamma-1} + \int_{\Omega} |\nabla v|^{2q} \\ & \leq C_6 \int_{\Omega} (u+1)^{p+\alpha+2\beta-2} |\nabla v|^2 + C_7 \int_{\Omega} (u+1)^2 |\nabla v|^{2q-2} + C_1 + C_5 \end{aligned}$$

for all $t \in (0, T_{\max})$ with certain positive constants C_6 and C_7 . Since $\alpha + 2\beta < \gamma + 1$, by Young's inequality we can find C_8 and C_9 such that

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{p} \int_{\Omega} (u+1)^p + \frac{1}{2q} \int_{\Omega} |\nabla v|^{2q} \right) \\ & + \frac{2M_1(p-1)}{(p-\alpha)^2} \int_{\Omega} |\nabla(u+1)^{\frac{p-\alpha}{2}}|^2 + \frac{q-1}{q^2} \| \nabla |\nabla v|^q \|_{L^2(\Omega)}^2 \\ & + \frac{\mu}{2^{\gamma+1}} \int_{\Omega} (u+1)^{p+\gamma-1} + \int_{\Omega} |\nabla v|^{2q} \\ & \leq \frac{\mu}{2^{\gamma+2}} \int_{\Omega} (u+1)^{p+\gamma-1} + C_8 \int_{\Omega} |\nabla v|^{\theta_1} + C_8 \int_{\Omega} |\nabla v|^{\theta_2} + C_9, \end{aligned} \quad (3.11)$$

where θ_1 and θ_2 are given by (2.8) and (2.9), respectively. According to the Gagliardo-Nirenberg inequality, for $i = 1, 2$ we can pick $C_{10} > 0$ such that

$$\begin{aligned} C_8 \int_{\Omega} |\nabla v|^{\theta_i} &= \| |\nabla v|^q \|_{L^{\frac{\theta_i}{q}}(\Omega)}^{\frac{\theta_i}{q}} \\ &\leq C_{10} \left(\| \nabla |\nabla v|^q \|_{L^2(\Omega)}^{\kappa_i} \| |\nabla v|^q \|_{L^{\frac{s}{q}}(\Omega)}^{1-\kappa_i} + \| |\nabla v|^q \|_{L^{\frac{s}{q}}(\Omega)}^{\frac{s}{q}} \right)^{\frac{\theta_i}{q}} \end{aligned} \quad (3.12)$$

for all $t \in (0, T_{\max})$, where κ_i is defined by (2.10). If $1 < \gamma < 2$, we choose $s = \frac{n}{n-1}$ in (3.12). Due to (2.2), (2.11) and the Young's inequality show that with some $C_{11} > 0$ we have for $i = 1, 2$

$$C_8 \int_{\Omega} |\nabla v|^{\theta_i} \leq \frac{q-1}{4q^2} \| \nabla |\nabla v|^q \|_{L^2(\Omega)}^2 + C_{11} \quad \text{for all } t \in (0, T_{\max}). \quad (3.13)$$

If $\gamma \geq 2$, we set $s = 2$ in (3.12). Then in view of (2.3), (2.12) and the Young's inequality, we obtain C_{12} such that

$$C_8 \int_{\Omega} |\nabla v|^{\theta_i} \leq \frac{q-1}{4q^2} \|\nabla |\nabla v|^q\|_{L^2(\Omega)}^2 + C_{12} \quad \text{for all } t \in (0, T_{\max}) \quad (3.14)$$

for $i = 1, 2$. Set

$$y(t) := \frac{1}{p} \int_{\Omega} (u+1)^p + \frac{1}{2q} \int_{\Omega} |\nabla v|^{2q}.$$

According to Young's inequality we can find C_{13} such that

$$\frac{\mu}{2\gamma+2} \int_{\Omega} (u+1)^{p+\gamma-1} \geq \int_{\Omega} (u+1)^p - C_{13} \quad \text{for all } t \in (0, T_{\max}) \quad (3.15)$$

and therefore in conjunction with (3.11), (3.13) and (3.14) and (3.15) we see that

$$y'(t) + C_{14}y(t) \leq C_{15} \quad \text{for all } t \in (0, T_{\max}) \quad (3.16)$$

with positive constants C_{14} and C_{15} . This completes the proof. \square

We can now pass to the proof of our main result.

Proof of Theorem 1.1. With the aid of Lemma 3.1, [16, Lemma A.1] and Lemma 2.1, we obtain the boundedness of u . Then the boundedness of v is based on a standard argument involving the variation-of-constants representation for v and the smoothing estimates for heat semigroup. \square

REFERENCES

- [1] X. CAO, *Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with logistic source*, J. Math. Anal. Appl., 412 (2014), pp. 181–188.
- [2] T. CIEŚLAK AND C. STINNER, *New critical exponents in a fully parabolic quasilinear keller-segel system and applications to volume filling models*, Journal of Differential Equations, 258 (2015), pp. 2080–2113.
- [3] T. HILLEN AND K. PAINTER, *Global existence for a parabolic chemotaxis model with prevention of overcrowding*, Adv. in Appl. Math., 26 (2001), pp. 280–301.
- [4] D. HORSTMANN AND G. WANG, *Blow-up in a chemotaxis model without symmetry assumptions*, European J. Appl. Math., 12 (2001), pp. 159–177.
- [5] D. HORSTMANN AND M. WINKLER, *Boundedness vs. blow-up in a chemotaxis system*, J. Differential Equations, 215 (2005), pp. 52–107.
- [6] S. ISHIDA, K. SEKI, AND T. YOKOTA, *Boundedness in quasilinear Keller-Segel systems of parabolic-parabolic type on non-convex bounded domains*, J. Differential Equations, 256 (2014), pp. 2993–3010.
- [7] E. F. KELLER AND L. A. SEGEL, *Initiation of slime mold aggregation viewed as an instability*, Journal of Theoretical Biology, 26 (1970), pp. 399–415.
- [8] J. LANKEIT, *Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source*, J. Differential Equations, 258 (2015), pp. 1158–1191.
- [9] X. LI AND Z. XIANG, *Boundedness in quasilinear keller-segel equations with nonlinear sensitivity and logistic source*, Discrete Contin. Dyn. Syst., 35 (2015), pp. 3503–3531.
- [10] N. MIZOGUCHI AND P. SOUPLET, *Nondegeneracy of blow-up points for the parabolic Keller-Segel system*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 31 (2014), pp. 851–875.
- [11] T. NAGAI, T. SENBA, AND K. YOSHIDA, *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis*, Funkcial. Ekvac., 40 (1997), pp. 411–433.
- [12] K. OSAKI, T. TSUJIKAWA, A. YAGI, AND M. MIMURA, *Exponential attractor for a chemotaxis-growth system of equations*, Nonlinear Anal., 51 (2002), pp. 119–144.
- [13] K. OSAKI AND A. YAGI, *Finite dimensional attractor for one-dimensional Keller-Segel equations*, Funkcial. Ekvac., 44 (2001), pp. 441–469.
- [14] K. J. PAINTER AND T. HILLEN, *Volume-filling and quorum-sensing in models for chemosensitive movement*, Can. Appl. Math. Q., 10 (2002), pp. 501–543.
- [15] P. QUITTNER AND P. SOUPLET, *Superlinear parabolic problems*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 2007. Blow-up, global existence and steady states.
- [16] Y. TAO AND M. WINKLER, *Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity*, J. Differential Equations, 252 (2012), pp. 692–715.

- [17] L. WANG, Y. LI, AND C. MU, *Boundedness in a parabolic-parabolic quasilinear chemotaxis system with logistic source*, Discrete Contin. Dyn. Syst., 34 (2014), pp. 789–802.
- [18] M. WINKLER, *Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source*, Comm. Partial Differential Equations, 35 (2010), pp. 1516–1537.
- [19] ———, *Does a ‘volume-filling effect’ always prevent chemotactic collapse?*, Math. Methods Appl. Sci., 33 (2010), pp. 12–24.
- [20] ———, *Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction*, J. Math. Anal. Appl., 384 (2011), pp. 261–272.
- [21] ———, *Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system*, J. Math. Pures Appl. (9), 100 (2013), pp. 748–767.
- [22] ———, *Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with strong logistic dampening*, J. Differential Equations, 257 (2014), pp. 1056–1077.
- [23] M. WINKLER AND K. C. DJIE, *Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect*, Nonlinear Anal., 72 (2010), pp. 1044–1064.
- [24] T. XIANG, *Boundedness and global existence in the higher-dimensional parabolic-parabolic chemotaxis system with/without growth source*, Journal of Differential Equations, (2015).

DEPARTMENT OF MATHEMATICS, SOUTHEAST UNIVERSITY, NANJING 211189, P. R. CHINA
E-mail address: qingshan11@yeah.net

DEPARTMENT OF MATHEMATICS, SOUTHEAST UNIVERSITY, NANJING 211189, P. R. CHINA
E-mail address: lieyx@seu.edu.cn